Generalized hierarchical model of defect development and self-organized criticality

M. G. Shnirman and Y. A. Tyurina
International Institute for Earthquake Prediction Theory and Mathematical Geophysics, Moscow, Russia
(Received 12 September 1996; revised manuscript received 9 June 1997)

A hierarchical model of defect development is found to display self-organized criticality (SOC) for a wide range of parameters. The model displays three different types of behavior, of which two are SOC behaviors. One of them had been found before in similar hierarchical systems, the other type being new. We show the difference between the two kinds and specify how the system goes from one SOC behavior to another while structure parameters of the system change discretely. The new type of SOC behavior allowed us to obtain a delocalization phenomenon for the model. The magnitude-frequency relation for both types of SOC behavior is approximately a straight line with a slope of -1. [S1063-651X(98)11403-4]

PACS number(s): 64.60.Lx

I. INTRODUCTION

One can understand self-organized criticality (SOC) in a dynamical system intuitively as a power law of its magnitude-frequency relation (the dependence of event number on event size) obtained in a general case [1]. (In fact, in [1] SOC means the independence of this power law form on the initial condition only.) If the system depends on a parameter, it is natural to call SOC a self-similarity of the magnitude-frequency relation for a wide range of the parameter [2]. That essentially differs from systems entering the critical state for one value of system parameter, when the magnitude-frequency relation that is exponential for an interval of parameters changes to a power law for one parameter value. An example of such behavior is phase transition in thermodynamics. In contrast to this, self-organized criticality (SOC) allows one to obtain a power law for the magnitude-frequency relation in a wide range of parameters instead of one isolated value.

A sandpile model displaying self-organized criticality was introduced in the well-known paper of Bak *et al.* [1]. Further investigations in this area showed other models with the same interesting feature, for example, [3] (linear spring model) and [4] (two-dimensional net of blocks and springs). Later the models were constructed expressing not only such natural phenomena as earthquakes [5] or solar radiation flow [6], but also such complexities as the behavior of biological systems [7], dynamics of economic indexes [8], or the occurrence of traffic jams [9] (see also review [10]). Despite such wide distribution of the self-organized criticality phenomenon, the exact reasons for its appearance are not revealed yet, and it is not possible to predict whether a system possesses this feature.

Global lithosphere activity is an important example of a physical process in which SOC is present. The Gutenberg-Richter law [11] expresses the magnitude-frequency relation for earthquakes as

$$\log N = a - bM,$$

where N is the number of earthquakes with magnitudes equal to or greater than M. This power law holds for a broad magnitude range. The Gutenberg-Richter law reflects the fact that the global distribution of earthquakes possesses the feature of self-organized criticality.

We are interested in dynamical models displaying SOC similar to the one expressed in the Gutenberg-Richter law. One type of system possessing this interesting feature consists of hierarchical models. A hierarchical model with branch number 3 was investigated in [2] and displayed self-organized criticality for a wide range of parameters. Our model is a generalization of the previous one. Increasing the branch number and introducing new threshold parameters, we have obtained a new kind of model behavior and called it ''unstable self-organized criticality.'' This kind of critical behavior has not been found previously. We also have obtained a critical behavior similar to the one observed in [2], which can be called ''stable self-organized criticality.'' We have described the new behavior and the difference between this and stable SOC behavior. We outlined how the transfer happens from known behavior to unknown. We also computed the magnitude-frequency relations for all kinds of system behavior and showed that for stable SOC and unstable SOC the magnitude-frequency law is self-similar.

The delocalization phenomenon is a deviation of the magnitude-frequency law from the straight line, expressed as an upward bend or gap in the line, or both. In case of seismicity models, delocalization implies that the number of observed earthquakes with large magnitudes is disproportionally large comparing with the number of earthquakes with small and medium magnitudes (upward bend) or an absence of earthquakes with magnitudes in some range (gap). Delocalization was obtained by Carlson *et al.* [12]. In their study of the Burridge-Knopoff model [13] of block and spring models, they obtained that small events are distributed in correspondence with the Gutenberg-Richter law, while large events exceeded the estimated number. Later Lomnitz-Adler *et al.* [14] obtained a delocalization phenomenon for models in which the stress is distributed among the nodes of a two-dimensional lattice, and Shnirman *et al.* [15] observed it by modeling an earthquake fault by means of a one-dimensional system of cellular automata. In our case we have specifically defined the event probability, which allowed us to observe a downward bend and gap in the magnitude-frequency relation for the case of unstable SOC behavior.

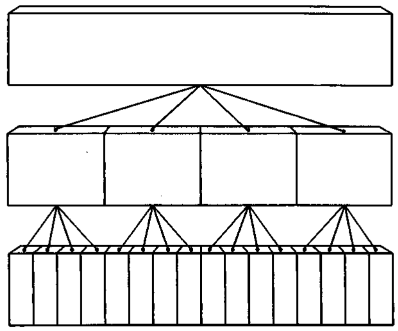

FIG. 1. A hierarchical tree with branch number $n=4$. A similar tree can be drawn for an arbitrary n.

In the second section of this paper a general description of the system is given and kinetic equations defining its evolution are written down. In the third section we show examples describing all types of system behavior and discuss their properties, depending on the values of system parameters. In the fourth section it is outlined how these types of behavior are situated in the two-dimensional space of system parameters. In the fifth section we find the magnitude-frequency law and show its realizations for different kinds of system behavior. The sixth section is devoted to the delocalization phenomenon displayed by our system. The seventh section contains conclusions and final remarks, and the eighth acknowledgments.

II. GENERAL DESCRIPTION OF THE MODEL

The model is constructed as an inverse cascade. We consider a hierarchical tree with branch number n (see Fig. 1). We denote the number of levels as $L+1$, and the number of current level as l. Numbers of level increase upward, the bottom level having the number 0, and the top level number L. The lth level of the tree contains an^{L-l} elements. For example, the bottom level (0th) contains an^L elements and the top level (Lth) contains a elements. In our computations we assume $a=1$. Any element of the $(l+1)$th level of the tree is connected by branches with n elements of the lower lth level. We call these n elements *a group of elements*. Any group of elements of the lth level corresponds to an element of the upper $(l+1)$th level.

Our model is a dynamic model, it evolves during time. The time increases discretely. At one time step the system passes from one state to another. A state of the system is defined by states of all its elements.

There are two possible states for any element of our model: the intact state and the defect state. At the initial time moment all elements are intact. Appearance of defect elements, or *defects*, at one time step obeys the following rules.

(i) At the bottom level, intact elements at the time moment t become defects with the probability α_0 at $t+1$.

(ii) At any other level l ($l=1,2,\dots,L$), the intact element at t becomes a defect at $t+1$ if the corresponding group of n elements of the $(l-1)$th level contained W defects or more at the moment t.

We call W *a functioning threshold*. We call the group of elements containing W defects or more *a critical configuration*. We call the group of elements containing fewer than W defects a *noncritical configuration*.

If a defect appears at the current time step we call it *a new defect*. If it appears at some previous time step, we refer to it as *an old defect*.

Rules (i) and (ii) imply that with increasing time all elements become defects sooner or later. To avoid such triviality we introduce rules for transformation of a defect element into an intact element (we call it *healing*).

(a) *Deterministic (pattern) healing*: if a group of n elements contains K defects or more at the time moment t, any element of the group becomes intact at $t+1$. We call parameter K *a healing threshold*.

(b) *Stochastic (individual) healing*: any defect element at level l can possibly change from ruptured to intact with *the healing intensity* $\beta(l)$, which is the probability for a defect element on the lth level at the time moment t to become intact at the time moment $t+1$.

If a defect becomes intact at the current time step, we call it *a new intact element*. If it appeared at some previous time step, and has not been ruptured, we refer to it as *an old intact element*.

Both kinds of healing are only for defects existing for more than one time step. If a defect has appeared at the current time step, healing rules do not apply to it.

We specify the order of application for the rules as follows. At one time step, rule (a) (deterministic healing) acts first, and transforms defects of the critical configuration to intact elements that should stay unruptured at this time step. After that, only the defects contained in the noncritical configurations remain. Then, to old intact elements, rules (i) and (ii) of defect development apply, creating new defects that should stay unhealed at this time step. Combinations of old and new defects that have made critical configurations transfer the perturbation to the next level of hierarchy. At last, rule (b) (stochastic healing) works among the old defects, changing (probably) some of them into intact elements.

We assume probability $\beta(l)$ depending on level number l and independent of time, namely,

$$\beta(l)=\beta_0 c^l,$$

where β_0 and c are constants in the range $0\leqslant\beta_0<1$ and $0<c<1$.

So, our system depends on three structure parameters: branch number n and thresholds K and W. Fixing those, we obtain a specified model and can investigate its behavior depending on evolution parameters α_0, β_0, and c. Changing those, we obtain different kinds of SOC, as shown below.

Let time increase discretely: $t=0,1,2,\dots$, structure parameters n, K, and W being fixed, and let our model evolve due to rules (i) and (ii) and (a) and (b). At the 0th level new defects would appear and would form critical configurations. These configurations would transport perturbation to the next level. Any new defect possibly will be healed. We compute states of all elements at every time step.

For the description of our model behavior we use the following values: (i) the intensity of defect generation on the lth level at moment t, $\alpha(l,t)$; (ii) healing intensity of single defects $\beta(l)$; (iii) density of groups of defects $P^i(l,t)$, that is, the probability for a group at level l and at time moment t to contain exactly i defects, $i=0,1,\dots,n$. Configurations containing equal numbers of defects have equal probabilities. Formulas for $\alpha(l,t)$ and $P^i(l,t)$ will be given below; (iv) the density of defect elements of level l at moment t denoted $p(l,t)$, that is, the mathematical expectation of the defect number in a group of defects at the lth level at time moment t divided by the number of elements in the group:

$$p(l,t)=\frac{1}{n}\sum_{i=1}^{i=n} i\binom{i}{n}P^i(l,t).$$

To obtain kinetic equations for our model, we express the probability for a group of elements of the lth level to contain exactly j defects at moment $t+1$ via probabilities for a group of elements of the lth level to contain exactly i defects at moment t and the conditional probabilities of passage from a configuration with i defects to one with j defects. We express it as

$$P^j(l,t+1)=\sum_{i=0}^{i\leqslant j,i<K} P^i(l,t)Q(i,j)+\sum_{i=j+1}^{i<K} P^i(l,t)R(i,j)+\sum_{i=K}^{n-j} P^i(l,t)S(i,j). \tag{1}$$

Here, $Q(i,j)$ denotes the conditional probability to pass from all configurations with i defects to the configuration with j defects, where $i<j$. At one time step, h defects from existing i defects stochastically heal, $i-h$ defects remain, $(j-i)+h$ new defects appear, and $(N-j)-h$ elements stay unruptured. So we obtain

$$Q(i,j)=\sum_{h=0}^{h\leqslant i,h\leqslant n-j}\binom{i-h}{j}[1-\beta(l)]^{i-h}\alpha(l,t)^{(j-i)+h}\times\binom{h}{n-j}\beta(l)^h[1-\alpha(l,t)]^{(n-j)-h}.$$

Similarly, $R(i,j)$ denotes the conditional probability to pass from all configurations with i defects to the configuration with j defects, where $j<i<K$: at one time step, $i-h$ defects from i defects stochastically heal, h defects remain, $j-h$ new defects appear, and $(N-j)-(i-h)$ elements stay unruptured. So

$$R(i,j)=\sum_{h=j-m}^{j}\binom{h}{j}[1-\beta(l)]^h\alpha(l,t)^{j-h}\binom{i-h}{n-j}\times\beta^{i-h}[1-\alpha(l,t)]^{(n-j)-(i-h)},$$

where $m=\min(j,n-i)$.

At last, $S(i,j)$ denotes the conditional probability to obtain a configuration with j defects from all possible configurations with i defects while $i\geqslant K$: at the time step these i defects heal deterministically, among other $N-i$ elements that were intact, j new defects appear, and $N-i-j$ defects remain intact. Hence

$$S(i,j)=\binom{i}{n-j}\alpha(l,t)^j[1-\alpha(l,t)]^{n-i-j}.$$

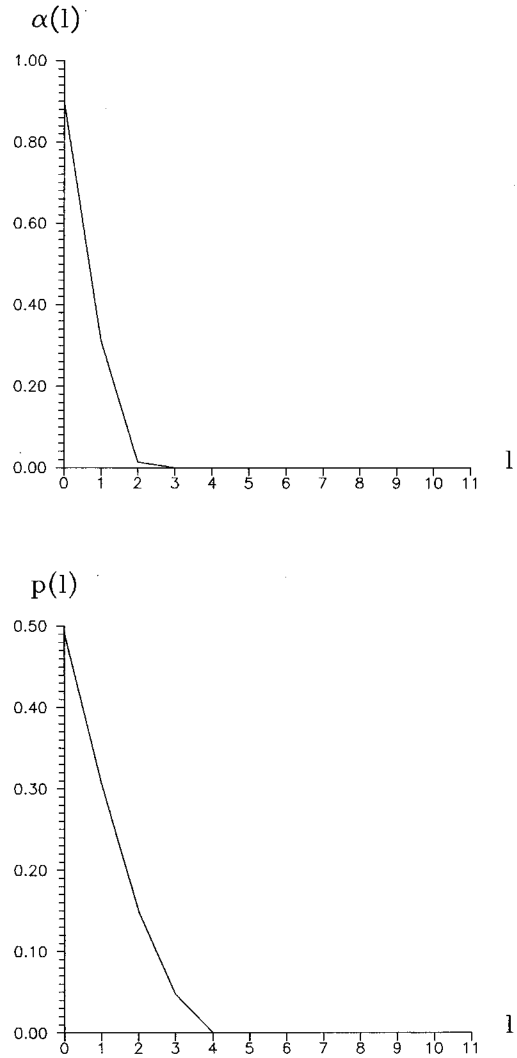

FIG. 2. An example of the stable behavior: functions $\alpha(l)$ and $p(l)$ for $n=4$, $K=2$, $W=4$, $\alpha_0=0.9$, $\beta_0=0$.

For the bottom 0th level the intensity of the appearance of new defects is constant. We denoted it above as α_0. The intensity of the generation of defects for the lth level is assumed to depend on critical configurations of the lth level. We express it as a sum of the probabilities $P^i(l,t)$ multiplied by the conditional probabilities to proceed from configurations with i defects to a critical configuration with j defects:

$$\alpha(l+1,t)=\sum_{j=W}^{n}\binom{j}{n}\left[\sum_{i=0}^{i<j,i<K} P^i(l,t)\binom{i}{j}\alpha(l,t)^{j-i}\times[1-\alpha(l,t)]^{n-j}+\sum_{i=K}^{n-j} P^i(l,t)\binom{i}{n-j}\times\alpha(l,t)^j[1-\alpha(l,t)]^{n-i-j}\right]. \tag{2}$$

The first part of the expression in large square brackets corresponds to noncritical configurations with i defects, in which $j-i$ new defects should appear. The second part of the expression corresponds to critical configurations with i

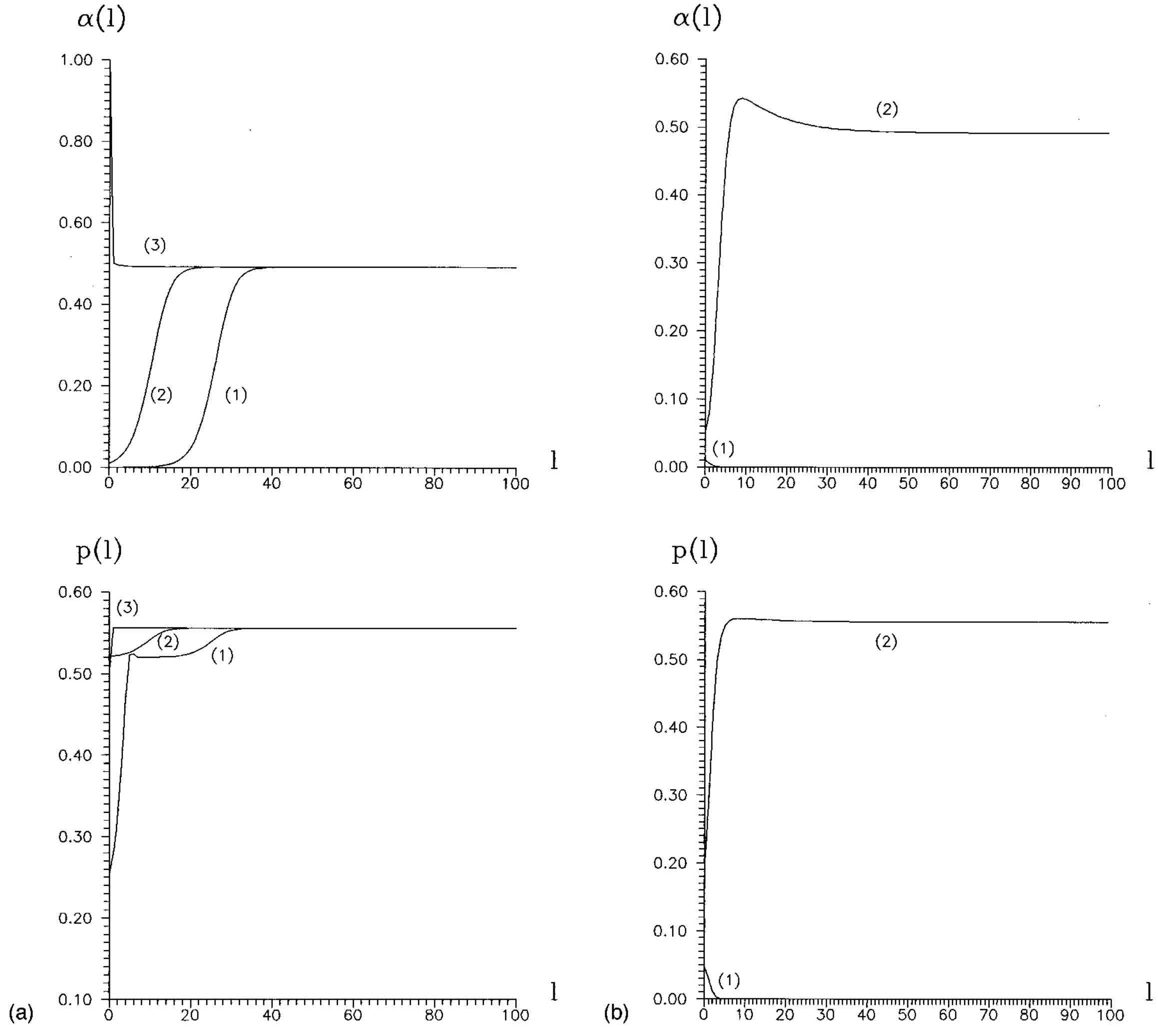

FIG. 3. (a) An example of stable SOC behavior for $\beta_0=0$: $n=4$, $K=4$, $W=2$. Line (1) corresponds to $\alpha_0=0.000\,01$, line (2) corresponds to $\alpha_0=0.01$, line (3) corresponds to $\alpha_0=0.999$. (b) An example of stable SOC behavior for $\beta_0>0$: $n=4$, $K=4$, $W=2$, $\beta_0=0.2$, $c=0.9$. Line (1) corresponds to subcritical $\alpha_0=0.01$, and almost coincides with the x axis; line (2) corresponds to overcritical $\alpha_0=0.05$.

defects that should be healed by deterministic healing, and among $n-i$ intact elements left, j elements should be changed to new defects.

Formulas (1) and (2) allow us to investigate the system behavior for different values of parameters α_0, β_0, and c while n, K, and W are fixed. We investigate the defect density $p(l,t)$ and the intensity of defect generation $\alpha(l,t)$ in order to construct the magnitude-frequency law on this basis. We use the other quantities for computation of these two.

All results described here were obtained by numerical modeling. The range of structure parameters is $3\leqslant n\leqslant 9$, $1\leqslant K\leqslant n$, $1\leqslant W\leqslant n$; that of evolution parameters is $0.001\leqslant\alpha_0\leqslant 0.999$, $0\leqslant\beta_0\leqslant 0.9999$, $0.7\leqslant c\leqslant 1$, unless otherwise specified.

III. THREE TYPES OF SYSTEM BEHAVIOR

We show four examples in order to describe the following three types of the system behavior.

First type. We use structure parameter values $n=4$, $K=2$, $W=4$ and obtain that at $t\rightarrow\infty$, $p(l,t)$ and $\alpha(l,t)$ for fixed l tend to the finite limits $p(l)$ and $\alpha(l)$. Functions $p(l)$ and $\alpha(l)$ decrease rapidly for increasing l, and for large l one has $p(l)=0$ and $\alpha(l)=0$ (see Fig. 2). This result was observed for any values of α_0, β_0, and c for given n, K, and W.

We call such behavior [when $p(l)$ and $\alpha(l)$ tend to 0] *stability*. For it, events of medium or large size are absent. Below, when we describe behavior of the system as stability, it means that $p(l,t)$ and $\alpha(l,t)$ decrease similarly, as shown in the above example, for a wide range of parameters α_0, β_0, and c.

Second type. We use parameter values $n=5$, $K=5$, $W=3$ and obtain that at $t\rightarrow\infty$, $p(l,t)$ and $\alpha(l,t)$ for fixed l tend to the finite limits $p(l)$ and $\alpha(l)$. While l increases, $p(l)$ tends to the asymptote p^*. Similarly, while l increases, function $\alpha(l)$ tends to the limit value α^*.

For $\beta_0=0$, considering any value of intensity of defect generation α_0, at $t\rightarrow\infty$ while l increases $p(l)$ tends to the stationary value $p^*>0$ and $\alpha(l)$ tends to the stationary value $\alpha^*>0$ [see Fig. 3(a)].

For $\beta_0>0$ there is a critical value $\alpha_{\rm cr}$ depending on β_0. For any $\alpha_0>\alpha_{\rm cr}$ at $t\rightarrow\infty$, while l increases, $p(l)$ tends to the

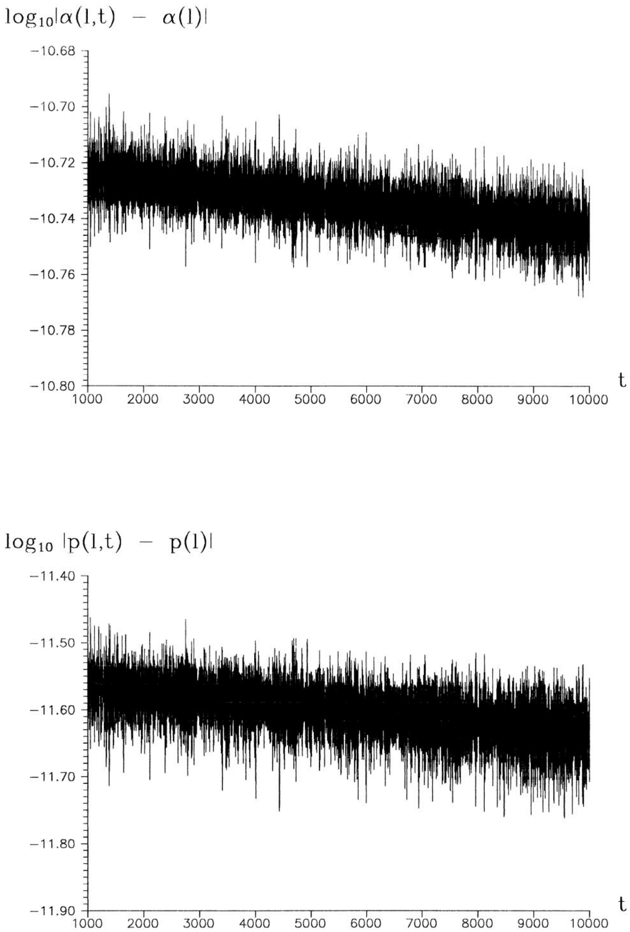

FIG. 4. Convergence of $p(l,t)$ and $\alpha(l,t)$ to their limits $p(l)$ and $\alpha(l)$, for $n=3$, $K=2$, $W=2$, $\alpha_0=0.2$, $\beta_0=0.2$, $c=0.9$, $t=1,\ldots,2000$.

value $p^*>0$ and $\alpha(l)$ to the value $\alpha^*>0$.

We call such behavior *stable self-organized criticality*. Below, when we call a system behavior the stable SOC, we mean that limits $p(l)$ and $\alpha(l)$ exist and tend to nonzero values p^* and α^* for a wide range of evolution parameters α_0, β_0, and c, while structure parameters n, K, and W are fixed.

In this example, for any $\alpha_0<\alpha_{\rm cr}$ at $t\to\infty$, while l increases, $p(l)$ and $\alpha(l)$ both decrease rapidly and for large l are equal to 0. So, for $\alpha_0<\alpha_{\rm cr}$ we observe stability, and for $\alpha_0>\alpha_{\rm cr}$ we observe the stable SOC [see Fig. 3(b)].

To find out how $p(l,t)$ and $\alpha(l,t)$ in this example converge to their limits $p(l)$ and $\alpha(l)$ as $t\to\infty$ for every l, we computed $\log_{10}|p(l,t)-p(l)|$ and $\log_{10}|\alpha(l,t)-\alpha(l)|$ for fixed l. Those quantities decrease, as is shown in Fig. 4.

Third type. The third type of behavior needs two examples for description.

For the first example we use structure parameters $n=3$, $K=2$, $W=1$. For any combination of α_0, β_0, and c, as $t\to\infty$, if we fix time step t (large enough), while l increases, $p(l,t)$ tends to the asymptote p^*. But as l increases further, $p(l,t)$ deviates from this asymptote and for different time steps tends to 0 or 1. If we fix level number l, then, as time goes on, for low levels $p(l,t)$ oscillates chaotically with small amplitude near the finite limit p^*, and for upper levels it oscillates chaotically with large amplitude. The intensity of defect development $\alpha(l,t)$ as $t\to\infty$, if we fix t and increase l, tends to the asymptote α^*, and, as l increases further, $\alpha(l,t)$ also deviates from it and tends to 0 or 1. If, for large enough fixed number of level l, at the moment t, $p(l,t)$ is close to 1, the corresponding $\alpha(l,t)$ is close to 0, and vice versa. For low levels, $\alpha(l,t)$ with an increase of t oscillates chaotically near the finite limit α^* (l is fixed), and the amplitude of this oscillation is small. For higher levels it oscillates chaotically with large amplitude. Graphs for $p(l,t)$ and $\alpha(l,t)$ are shown in Fig. 5(a). An example of oscillation at a fixed level is shown in Fig. 6.

We call such behavior *unstable self-organized criticality*. Below, when we call a system behavior the unstable SOC, we mean that the quantities $p(l,t)$ and $\alpha(l,t)$ change with time steps as was described in this example, for a wide range of evolution parameters α_0, β_0, and c (while structure parameters n, K, and W are fixed).

To our knowledge, this type of behavior is entirely new. It has not been observed in previous works concerned with hierarchical systems.

Despite the oscillation and apparent chaos of the behavior, its magnitude-frequency relation shown below allowed us to refer to it as another manifestation of SOC in an evidently chaotic system. This is also true for the next example.

The second example takes into account structure parameters $n=4$, $K=2$, $W=2$.

For $\beta_0=0$ and any α_0 the system behavior copies the previous example and displays unstable SOC. For the case $\beta_0>0$ there exists a critical value $\alpha_{\rm cr}$ such that for any $\alpha_0<\alpha_{\rm cr}$ as $t\to\infty$, while l increases, $p(l,t)$ and $\alpha(l,t)$ decrease rapidly and are 0 for large l, so for $\alpha_0<\alpha_{\rm cr}$ we observe stability here. For any $\alpha_0>\alpha_{\rm cr}$, $p(l,t)$ and $\alpha(l,t)$ are as in the case of $\beta_0=0$: while l increases, they tend to asymptotes p^* and α^*, and then alternate between 0 and 1 [see Fig. 5(b)]. Oscillation occurs similar to that described in the previous example. This behavior is the unstable SOC.

On the whole, for a fixed combination of structure parameters n, K, and W, the system displays stability for any α_0, β_0, and c, stable SOC for one range of evolution parameters and stability for another, unstable SOC for any α_0, β_0, and c, or unstable SOC for one range of evolution parameters and stability for another.

IV. THE ARRANGEMENT OF THE TYPES OF SYSTEM BEHAVIOR IN THE TWO-DIMENSIONAL PARAMETER SPACE

The system displays three types of behavior, depending on the combination of structure parameters K and W, which may vary from 1 to n. For any pair (K,W) the system behavior can be pointed out.

Let us show the dependence of the system behavior on the pair (K,W) of threshold parameters for $n=6$ as a table. Here a ‘‘stability’’ square means stable behavior for the entire range of evolution parameters; a ‘‘stable SOC’’ square means that for this (K,W) pair, stable SOC and stability are observed. An ‘‘unstable SOC’’ square means that for this (K,W) pair, unstable SOC behavior, or unstable SOC and stability, is observed.

Looking at table, one can see that the values of threshold parameters that produce self-organized critical behavior

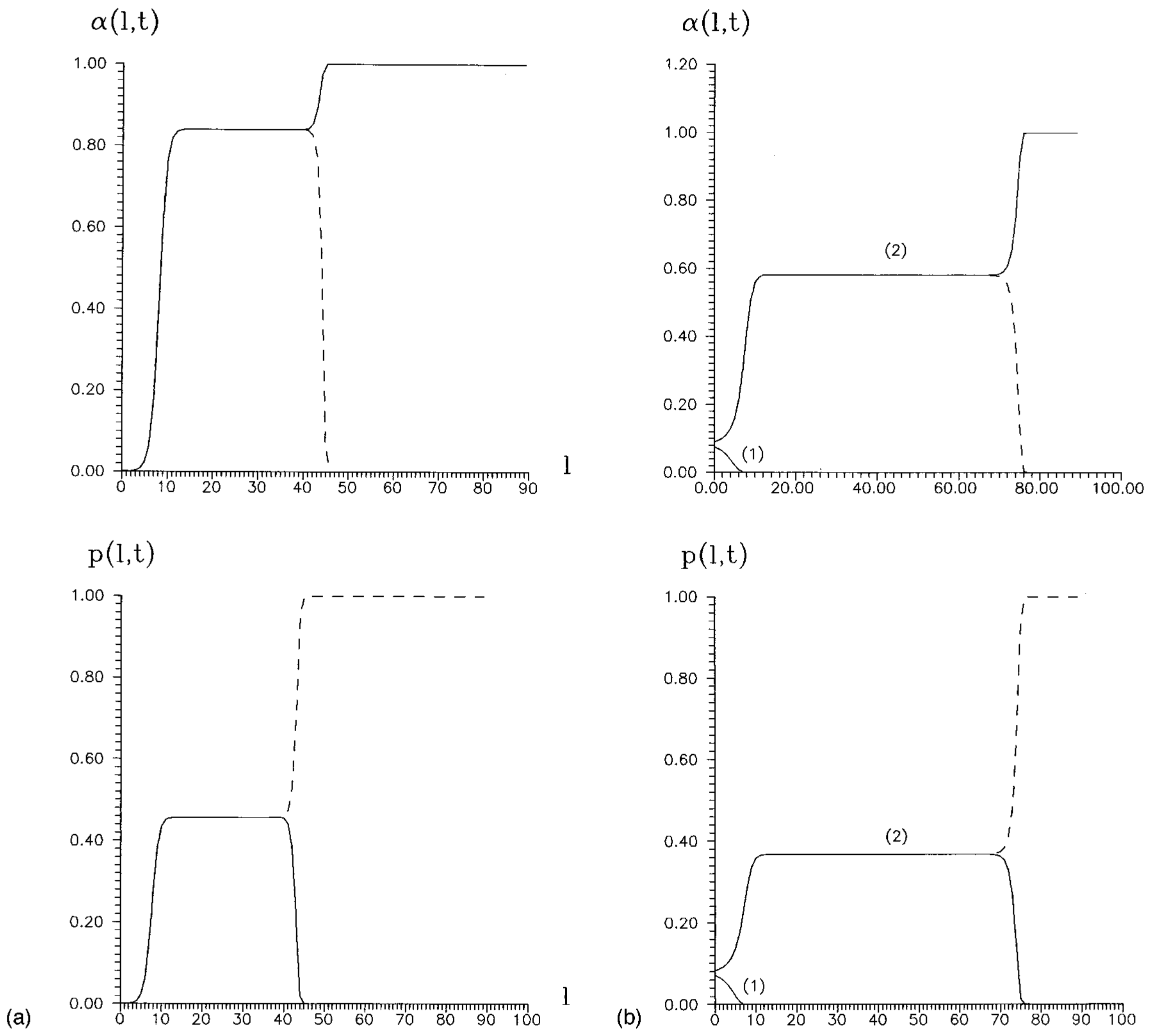

FIG. 5. (a) An example of unstable SOC for the case $W=1$: $n=3$, $K=2$, $\alpha_0=0.0001$, $\beta_0=0.9999$, $c=0.9$. Solid line is for $t=34\,999$, dashed for $t=35\,000$. (b) An example of unstable SOC for the case $W>1$, $\beta_0>0$: $n=4$, $K=2$, $W=2$, $\beta_0=0.99$, $c=1$. Line (1) corresponds to subcritical $\alpha_0=0.075$, line (2) corresponds to overcritical $\alpha_0=0.09$. Solid part of line (2) corresponds to the time moment $t=19\,999$, and dashed line to $t=20\,000$.

(stable and unstable) occupy a connected domain $K\geqslant W$ in the two-dimensional space of parameters, excluding the values of W close to n. In the table this domain is outlined by a dashed line. For any other pair of structure parameters our system displays stability, that is, for $K\geqslant W$, where W is close to n, and for all $K\leqslant W$.

The threshold parameters that produce self-organized critical behavior divide into two groups.

(a) The group of stable self-organized criticality consists of two parts. One part consists of pairs (K,W) with $K=n$ and any $W>1$ except W close to n (rightmost column of the table); the other contains one more pair (K,W), where $n>K\geqslant W$ and W is equal to the greatest W (or less 1) for pairs with $K=n$.

(b) The group of unstable self-organized criticality contains all the other pairs. For the pairs with $W=1$ the behavior is similar to that shown in the first example of the unstable SOC (the unstable SOC for the entire range of evolution parameters), and for pairs with $W>1$ the behavior is as in the second example (the unstable SOC for one range of evolution parameters, the stability for the other).

A similar table can be made for an arbitrary n: domains of stability and regions containing stable and unstable self-organized criticality are situated in the space of threshold parameters (K,W), $1\leqslant K\leqslant n$, $1\leqslant W\leqslant n$ as described above (see Fig. 7).

V. MAGNITUDE-FREQUENCY RELATION FOR THE SYSTEM

The magnitude-frequency relation displays the dependence of event number on event energy. We assume that the energy of an event depends on the number of the level of hierarchy:

$$E(l)=an^l,$$

where a is a constant giving the number of events for the highest level of hierarchy. In our computations we assumed $a=1$. The probability of the appearance of a new defect at the level l during time step from the moment t until $t+1$ is

$$P_{\text{new}}=\alpha(l,t)[1-p(l,t)].$$

To find the mean number of events for one time step $N(l,t)$, we multiply the number of elements of the level by the probability of the appearance of a new defect:

$$N(l,t)=P_{\mathrm{new}}n^{L-l}.$$

When our system displays stability or stable self-organized critical behavior, $\alpha(l,t)$ and $p(l,t)$ tend to the stationary values $\alpha(l)$ and $p(l)$ as $t\to\infty$, and we can find the mean number of events at the level l for one time step as

$$N(l)=\alpha(l)[1-p(l)]n^{L-l}.$$

In the case of stable self-organized criticality, the magnitude-frequency relation in the log-log scale is a line close enough to the straight line with the slope -1 (see Fig. 8). For the bottom levels (first 5–10 levels) the slope of the magnitude-frequency line is less than -1. As l increases, the slope tends to -1 and for the upper levels, where $\alpha(l)$ and $p(l)$ are already very close to the stationary values p^* and α^*, the slope becomes equal to -1. Figure 8 shows the magnitude-frequency relations for two different values of K. The magnitude-frequency lines for both cases are almost the same.

For $\beta_0=0$ and $\beta_0>0$ the magnitude-frequency relations differ. For $\beta_0=0$ and any α_0 the magnitude-frequency line is close enough (in the sense described above) to the straight line with the slope -1. For $\beta_0>0$ all α_0 divide into subcritical and overcritical. For subcritical α_0, $p(l)$ and $\alpha(l)$ vanish, and the magnitude-frequency relation decreases rapidly. For overcritical values, when $p(l)$ and $\alpha(l)$ tend to asymptotes, it is close to a straight line with the slope -1 (see Fig. 9).

When the system displays unstable self-organized critical behavior, there are no stationary values for $\alpha(l,t)$ and $p(l,t)$, so it seems natural to define a magnitude-frequency relation for a mean number of events on the time interval ΔT,

$$N_{\mathrm{mean}}(l)=\frac{1}{\Delta T}\sum_{t=T_0}^{T_0+\Delta T}N(l,t)$$

for large enough T_0 (its value depends on the convergence rate of the process and is defined by values of structure parameters n,K,W) and for large enough ΔT.

The magnitude-frequency line consists of two parts of a straight line with the slope -1 connected by a short (5–8

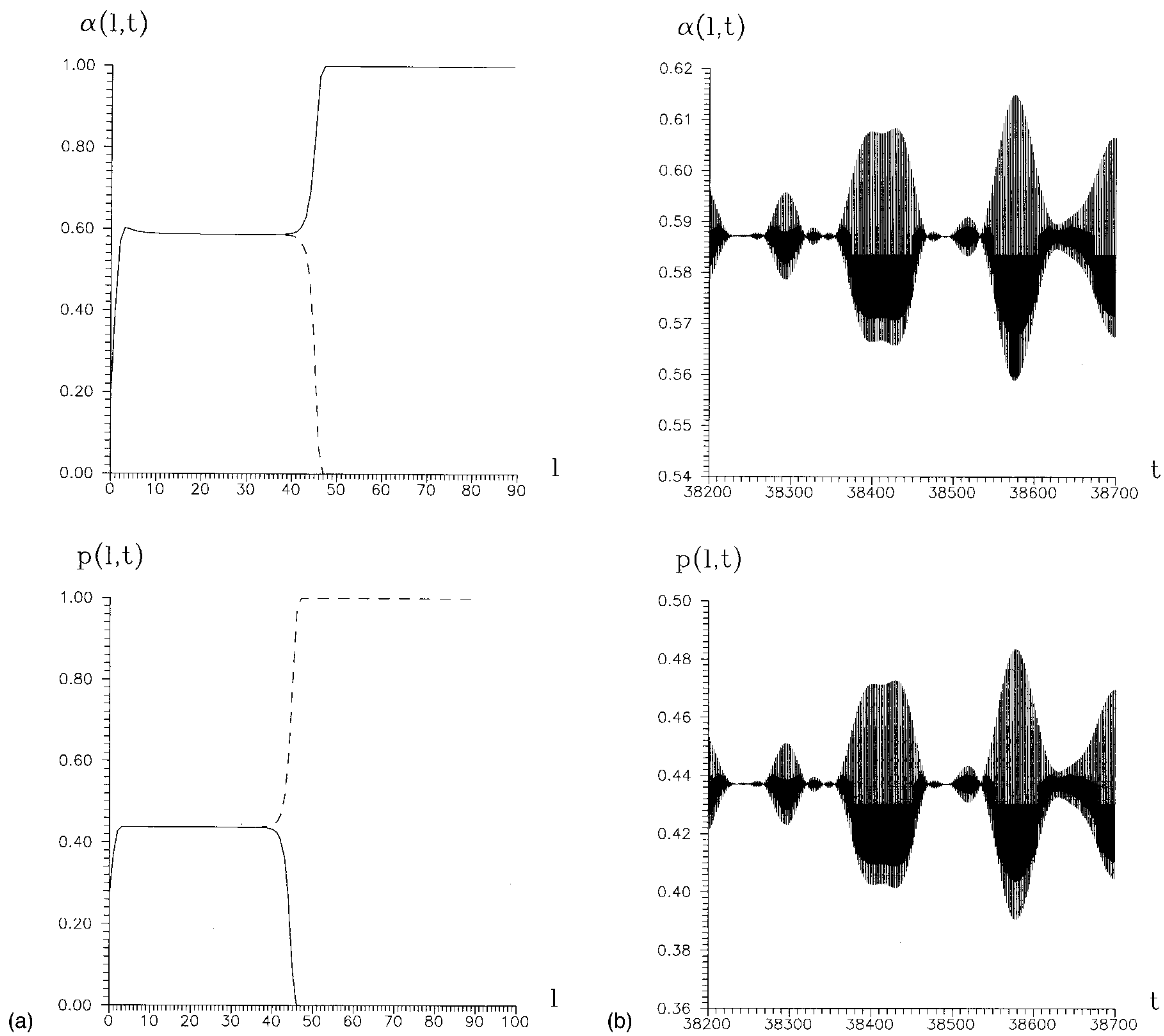

FIG. 6. (a) Graphs of $p(l,t)$ and $\alpha(l,t)$ for $n=5$, $K=3$, $W=2$, $t=39\,999{,}40\,000$. (b) Chaotic oscillation of $p(l,t)$ and $\alpha(l,t)$ for fixed level number $l=41$ and other parameters as in Fig. 5(a). (c) Part of (b) to larger scale.

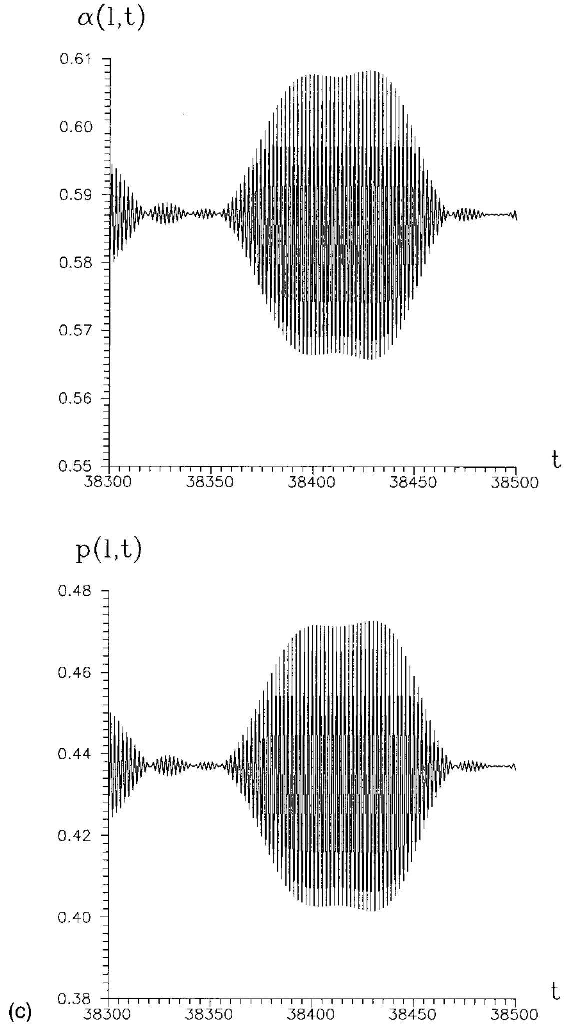

FIG. 6 (*Continued*).

TABLE I. The arrangement of the types of system behavior in two-dimensional parameter space for $n=6$.

W \ K	1	2	3	4	5	6
1	Unstable SOC	Unstable SOC	Unstable SOC	Unstable SOC	Unstable SOC	Stable SOC
2	Stabi-lity	Unstable SOC	Unstable SOC	Unstable SOC	Unstable SOC	Stable SOC
3	Stabi-lity	Stabi-lity	Unstable SOC	Unstable SOC	Unstable SOC	Stable SOC
4	Stabi-lity	Stabi-lity	Stabi-lity	Stable SOC	Unstable SOC	Stable SOC
5	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity
6	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity	Stabi-lity

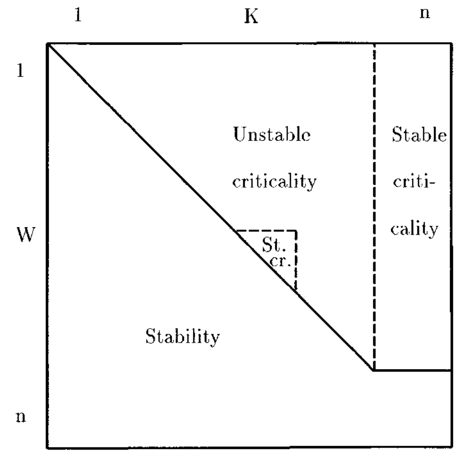

FIG. 7. The arrangement of stable and unstable SOC and stability areas in the two-dimensional parameter space (K,W) for an arbitrary n.

levels) line, corresponding to the movement of the system with the growth of l from asymptotes to the extreme points 0 and 1. This short line also is close to the straight line with the slope -1.

For the fixed n,K,W and changing α_0, magnitude-frequency lines have small differences in the initial levels and in the short lines connecting linear parts, as is shown in Fig. 10. When α_0 is changed, the magnitude-frequency line changes its slope at the beginning and (very slightly) in the middle of the line. Its linear parts for both cases coincide, with the slope -1.

At last, for the case of stability we obtain that the magnitude-frequency relation vanishes, as is shown in Fig. 11.

VI. THE DELOCALIZATION PHENOMENON

To obtain delocalization in our model, we consider the case of unstable SOC. A new defect, appearing at level l,

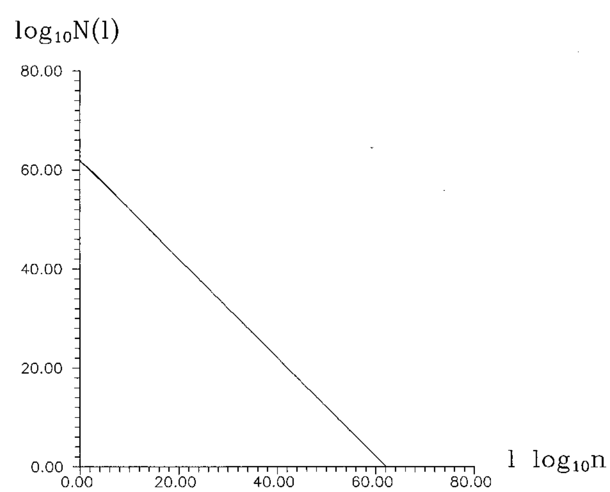

FIG. 8. Magnitude-frequency relations for stable SOC and two different values $K=3$ and $K=5$. The other parameters are $n=5$, $\alpha_0=0.15$, $\beta_0=0.2$, $c=0.9$, $W=3$. Magnitude-frequency lines for both cases almost coincide.

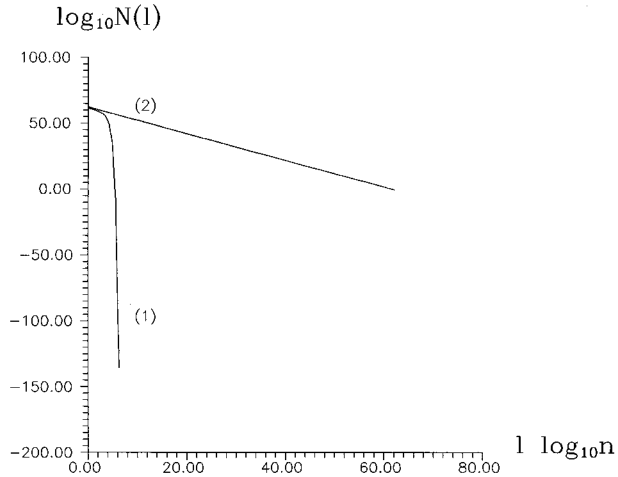

FIG. 9. Magnitude-frequency relations for stable SOC, parameters $n=5$, $K=5$, $W=3$, $\beta_0=0.2$, $c=0.9$, and two values of α_0, subcritical and overcritical. The magnitude-frequency relation in the first case [line (1) for subcritical $\alpha_0=0.05$] decreases rapidly. In the second case [line (2) for overcritical $\alpha_0=0.99$] the magnitude-frequency relation is a straight line with the slope of -1.

will probably become a part of the critical configuration, by which the perturbation is transferred to the next level $l+1$, thus forming a new defect. Here we will calculate the probability of the appearance of a defect of level l that does not lead to the transfer of perturbation to the next level $l+1$. This allows us to identify events of a given level, which do not participate in an event of the next level.

We denote as $P_{\text{new}}(l,t)$ the probability to have the critical configuration with at least one new (appearing during the last time step) defect. To obtain the exact formula for $P_{\text{new}}(l,t)$, we use this consideration. Let us denote as $P_m(l,t)$ the probability for the critical configuration to contain exactly m new defects. Then $P_m(l,t)$ consists of conditional probabilities to

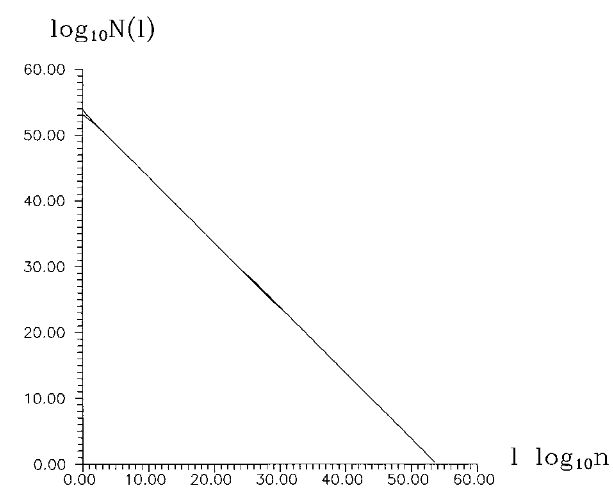

FIG. 10. Magnitude-frequency relations in the case of unstable SOC and two different $\alpha_0=0.1$ and $\alpha_0=0.9$. The other parameters $n=4$, $K=2$, $W=2$, $\beta_0=0.7$, $c=0.9$. Magnitude-frequency lines corresponding to different α_0 differ at the initial levels and in the middle of the plot. Linear parts of both lines coincide. The slope of these lines is -1.

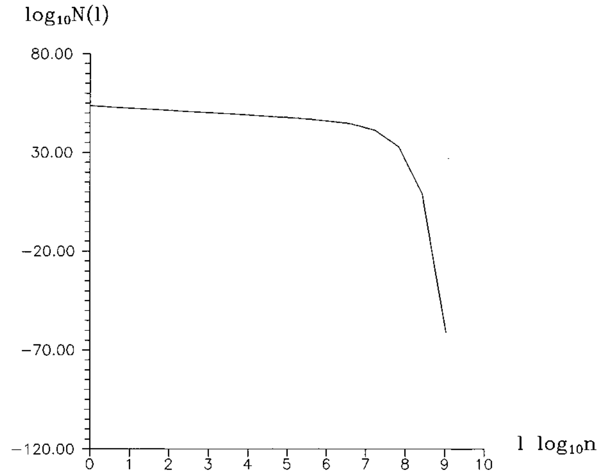

FIG. 11. An example of the magnitude-frequency relation for the case of stability. Parameter values are $n=4$, $K=3$, $W=3$, $\alpha_0=0.7$, $\beta_0=0.3$, $c=0.9$.

pass from all configurations with no less than m intact elements to a critical configuration, where exactly m of them have become the new defects:

$$P_m(l,t)=\sum_{i\geq 0, i\geq W-m}^{n-m}\binom{i}{n}P^i(l,t)\binom{m}{n-i}\alpha(l,t)^m[1-\alpha(l,t)]^{n-i-m}$$

and the complete probability for the new defect to appear within the critical configuration is

$$P_{\text{new}}(l,t)=\frac{1}{n}\sum_{m=1}^{n} mP_m(l,t).$$

Then the probability of the appearance of a new defect which does not lead to the transfer of perturbation to the next level, at the moment t and level l, is

$$P_a(l,t)=\alpha(l,t)[1-p(l,t)]-P_{\text{new}}(l,t)[1-p(l+1,t)], \tag{3}$$

where, as one can easily see, $\alpha(l,t)[1-p(l,t)]$ is the probability of the appearance of a new defect at level l, and $P_{\text{new}}(l,t)[1-p(l+1,t)]$ is the probability for this newborn defect to be part of the process of transfer of the perturbation to the next $l+1$ level. For the top level with the number L one has $P_a(L,t)=p(L,t)$, because the defects of higher levels do not exist.

The mean number of events at level l, $N(l,t)$, is the pure probability of the appearance of a defect multiplied by the number of elements at level

$$N(l,t)=P_a(l,t)n^{L-l}.$$

The magnitude-frequency line is straight for levels where $p(l,t)$ and $\alpha(l,t)$ tend to asymptotes. For levels where $p(l,t)$ and $\alpha(l,t)$ deviate from the asymptotes, the magnitude-frequency line bends downward. A gap on the line corresponds to levels where $p(l,t)$ and $\alpha(l,t)$ are equal to 0 or 1. For these levels of hierarchy events do not occur—

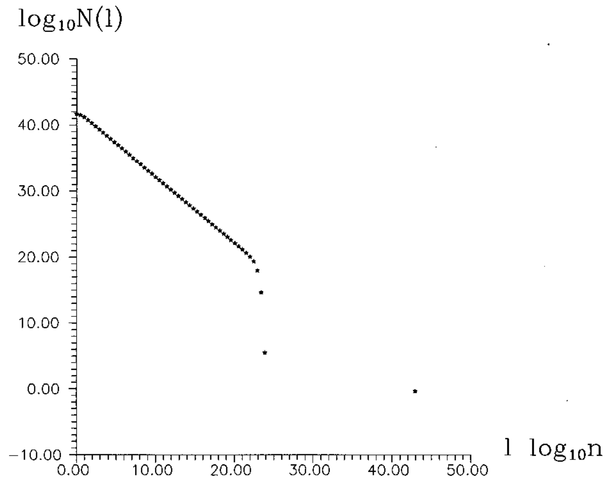

FIG. 12. Magnitude-frequency relation for ''pure probabilities'' displays a downward bend and a gap at the right side of the plot. For this plot $n=3$, $K=2$, $W=1$. The slope of the linear part is -1.001.

any event of this size implies the largest event and participates in it. And there is also an isolated point showing the number of events for the top level of hierarchy. An example of such a magnitude-frequency relation is shown in Fig. 12.

VII. CONCLUSIONS

We have constructed and investigated a hierarchical system that evolved in accordance with given kinetic equations and depended on two sets of parameters. We observed that two of the three types of behavior of this system display SOC for a wide range of parameters, while in the third case stability occurs. For the first behavior type, which we called stable SOC, the functions $p(l,t)$ and $\alpha(l,t)$, describing the system behavior, tend to finite limits as $t\rightarrow\infty$ for every l. This is similar to the system behavior observed and described before [2]. For the second type, which we called unstable SOC, functions $p(l,t)$ and $\alpha(l,t)$ for every l oscillate chaotically as t increases. The amplitude of oscillation depends on the level number. It is small for lower levels and large for higher ones. This chaotic behavior results from the large dimension of the phase space for each level.

To our knowledge, such behavior was not observed in previous works concerned with hierarchical systems and is entirely new. To define the attractor dimension of the system in such a chaotic case is an interesting problem.

We divided the two-dimensional space of the structure parameters (K,W) into areas corresponding to the different types of system behavior. That allowed us to define the dependence of the system behavior on the values of all system parameters: the pair (K,W) defines the type of behavior, while the evolution parameters α_0, β_0, and c define the development and behavior of the system within the given type in the way described above.

Magnitude-frequency relations were computed for the three types of system behavior. For stable SOC and unstable SOC the magnitude-frequency relation is approximately a straight line with a slope of -1. For the case of stability, the magnitude-frequency relation decreases to 0 for some first levels of the system and is equal to 0 for the other levels.

We introduced a special expression for event probabilities in order to distinguish events of small size, which do not participate in events of larger size, and to avoid counting one event, which lasts for some time steps, more than one time. The magnitude-frequency law, constructed on the basis of this expression, shows a big gap corresponding to events of large sizes. In the system, only events of small, medium, and maximum possible size may occur. Any large event is only a part of an event of the largest size. So, we obtained a delocalization phenomenon. It is in accordance with real observations, when for a given seismic region only small, medium, and largest (''characteristic'') earthquakes are observed, large events are absent [16].

ACKNOWLEDGMENTS

This work was partially supported by U.S. National Science Foundation Grant No. EAR 94 23818, and International Science and Technology Center, Moscow (Project No. 415-96).

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1987).
[2] E. Blanter and M. Shnirman, Phys. Rev. E **53**, 3408 (1996).
[3] J. Carlson and J. Langer, Phys. Rev. Lett. **62**, 2632 (1989).
[4] Z. Olami, H. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
[5] B. Barriere and D. Turcotte, Phys. Rev. E **49**, 1151 (1994).
[6] L. Vlahos, M. Georgoulis, R. Kluiving, and P. Paschos, Astron. Astrophys. **299**, 897 (1995).
[7] R. V. Sole, D. Lopez, M. Ginovart, and J. Valls, Phys. Lett. A **172**, 56 (1992).
[8] R. Mantegna and H. Stanley, Nature (London) **376**, 46 (1995).
[9] T. Natagani, J. Phys. Soc. Jpn. **64**, 31 (1995).
[10] K. Ito, PAGEOPH **138**, 531 (1992).
[11] B. Gutenberg and C. Richter, *Seismicity of the Earth and Associated Phenomena* (Princeton University Press, Princeton, 1949).
[12] J. Carlson, J. Langer, B. Shaw, and C. Tang, Phys. Rev. A **44**, 884 (1991).
[13] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).
[14] J. Lomnitz-Adler, L. Knopoff, and G. Martinez-Mekler, Phys. Rev. A **45**, 2211 (1992).
[15] M. G. Shnirman and J. V. Genyuk, Comput. Seismol. Geodynam. (to be published).
[16] D. P. Schwartz and K. J. Coppersmith, J. Geophys. Res. **89**, 5681 (1984).